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WINNING SETS OF REGULAR LANGUAGES: DESCRIPTIONAL AND COMPUTATIONAL COMPLEXITY

Pierre Marcus [A] Ilkka Törmä [B,C]

[A] *M2 informatique fondamentale, École Normale Supérieure de Lyon*
46, allée d'Italie, 69364 Lyon, France
`pierre.marcus@ens-lyon.fr`

[B] *Department of Mathematics and Statistics, University of Turku*
20014 Turku, Finland
`iatorm@utu.fi`

ABSTRACT

We investigate certain word-construction games with variable turn orders. In these games, Alice and Bob take turns on choosing consecutive letters of a word of fixed length, with Alice winning if the result lies in a predetermined target language. The turn orders that result in a win for Alice form a binary language that is regular whenever the target language is, and we prove some upper and lower bounds for its state complexity based on that of the target language. We also consider the computational complexity of membership and intersection problems of winning sets.

Keywords: state complexity, regular languages, winning sets

1. Introduction

Let us define a word-construction game of two players, Alice and Bob, as follows. Choose a set of binary words $L \subseteq \{0,1\}^*$ called the *target set*, a length $n \geq 0$ and a word $w \in \{A,B\}^n$ called the *turn order*, where A stands for Alice and B for Bob. The players construct a word $v \in \{0,1\}^n$ so that, for each $i = 0, 1, \ldots, n-1$ in this order, the player specified by w_i chooses the symbol v_i. If $v \in L$, then Alice wins the game, and otherwise Bob wins. The existence of a winning strategy for Alice depends on both the target set and the turn order. We fix the target set L and define its *winning set* $W(L)$ as the set of those words over $\{A,B\}$ that result in Alice having a winning strategy.

Winning sets were defined under this name in [13] in the context of symbolic dynamics, but they have been studied before that under the name of *order-shattering sets* in [1, 5]. The winning set has several interesting properties: it is downward closed

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Pierre Marcus: 0000-0002-0631-4284, Ilkka Törmä: 0000-0001-5541-8517