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# ASYMPTOTIC ABELIAN COMPLEXITIES OF CERTAIN MORPHIC BINARY WORDS

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ABSTRACT

We study asymptotic Abelian complexities of morphic binary words. We complete the classification of upper Abelian complexities of pure morphic binary words initiated recently by F. Blanchet-Sadri, N. Rampersad, and N. Fox. We also study a class of morphic binary words having different asymptotic factor complexities despite having the same asymptotic Abelian complexity.

*Keywords:* morphic words, Abelian complexity, factor complexity

## 1. Introduction

The study of complexity measures of infinite words is a well-motivated and actively studied research area. The *factor complexity function*

$$\mathcal{P}_w : \mathbb{N} \to \mathbb{N}$$

of an infinite word $w \in \Sigma^{\mathbb{N}}$ counts, for each $n \in \mathbb{N}$, the number of distinct factors of $w$ of length $n$. The notion is a fundamental one in combinatorics of infinite words. This can be seen, for instance, from the theorem of M. Morse and G.A. Hedlund [14], which characterises *ultimately periodic* words as exactly the words admitting $\mathcal{P}(n_0) \leq n_0$ for some $n_0 \in \mathbb{N}$. For surveys on factor complexity we refer the reader to [3, 4].

Inspired by the notion of factor complexity, other complexity measures have been developed. One such measure is the *Abelian complexity* of infinite words, the topic of this note. For other related complexity measures, see for instance [10, 17, 20]. Two finite words $u, v$ are said to be *Abelian equivalent*, denoted by $u \sim v$, if, for each letter $a$, the word $u$ contains equally many $a$'s as the word $v$. Note that the Abelian equivalence is an equivalence relation on words. The *Abelian complexity function*

$$\mathcal{P}_w^{\mathrm{ab}} : \mathbb{N} \to \mathbb{N}$$

of an infinite word $w$ then counts, for each $n$, the number of distinct Abelian equivalence classes of length $n$ occurring in the word $w$. (The subscript is omitted when $w$