Journal of Automata, Languages and Combinatorics **21** (2016) 3, 183 – 201

COUNTING OCCURRENCES OF A PATTERN OF LENGTH THREE WITH AT MOST TWO DISTINCT LETTERS IN A k-ARY WORD

Toufik Mansour[(A)] Armend Sh. Shabani[(B)]

[(A)] *University of Haifa, Department of Mathematics*
3498838 Haifa, Israel
tmansour@univ.haifa.ac.il

[(B)] *University of Prishtina, Department of Mathematics*
10000 Prishtinë, Republic of Kosova
armend.shabani@uni-pr.edu

ABSTRACT

Define $\tau(\pi)$ to be the number of subsequences of π that are order-isomorphic to τ. Let τ be a pattern of length three with at most two distinct letters, namely,

$$\tau \in \{111, 112, 121, 122, 211, 212, 221\}.$$

In this paper, we give an algorithm for finding the generating function

$$w_{\tau;r}(n;y) = \sum_{k\geq 1} \sum_{\pi\in[k]^n,\tau(\pi)=r} y^k$$

for the number of k-ary words of length n that contain exactly r occurrences of the pattern τ, for given $r \geq 0$. In particular, we obtain explicit formulas for the generating functions $w_{\tau;r}(n;y)$, where $r = 0, 1$.

Keywords: k-ary word, pattern, enumeration, generating function, Eulerian polynomial

1. Introduction

Permutations. We denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$ by S_n. We shall view permutations in S_n as words with n distinct letters in $[n]$. A *permutation pattern* or just *pattern* is a permutation $\tau \in S_\ell$, and an *occurrence* of τ in a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is a subsequence of π that is order-isomorphic to τ. For instance, an occurrence of 312 is a subsequence $\pi_a\pi_b\pi_c$ $(1 \leq a < b < c \leq n)$ of π such that $\pi_b < \pi_c < \pi_a$. We denote the number of permutations in S_n that contain exactly r occurrences of the pattern τ by $s_{\tau;r}(n)$. In the last two decades much attention has been paid to the problem of finding the numbers $s_{\tau;r}(n)$ for a fixed $r \geq 0$ and a given pattern τ (see [1,2,8,10–12] and references therein). Up to now, only the