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SEMIRING FRAMEWORKS AND ALGORITHMS FOR SHORTEST-DISTANCE PROBLEMS

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ABSTRACT

We define general algebraic frameworks for shortest-distance problems based on the structure of semirings. We give a generic algorithm for finding single-source shortest distances in a weighted directed graph when the weights satisfy the conditions of our general semiring framework. The same algorithm can be used to solve efficiently classical shortest paths problems or to find the k-shortest distances in a directed graph. It can be used to solve single-source shortest-distance problems in weighted directed acyclic graphs over any semiring. We examine several semirings and describe some specific instances of our generic algorithms to illustrate their use and compare them with existing methods and algorithms. The proof of the soundness of all algorithms is given in detail, including their pseudocode and a full analysis of their running time complexity.

Keywords: Semirings, finite automata, shortest-paths algorithms, rational power series

Introduction

Classical shortest-paths problems in a weighted directed graph arise in various contexts. The problems divide into two related categories: single-source shortest-paths problems and all-pairs shortest-paths problems. The single-source shortest-path problem in a directed graph consists of determining the shortest path from a fixed source vertex s to all other vertices. The all-pairs shortest-distance problem is that of finding the shortest paths between all pairs of vertices of a graph.

In the classical shortest-paths problems, edge weights may represent distances, costs, or any other real-valued quantity that can be added along a path, and that one wishes to minimize. Edge weights are real numbers (elements of $\mathbb{R}$), and the specific operations used are: addition ($+$) along a path, and minimum (min) applied to path weights.

Classical shortest-paths problems can be generalized to other weight sets, and to other operations. The weights, elements of a set $\mathbb{K}$, may be real numbers, strings, regular expressions, subsets of another set, or any other quantity that can be *multiplied* along a path using an operation $\otimes$, and that can be *summed* using an operation $\oplus$. The weight of a path is obtained by multiplying edge weights along that path using $\otimes$,