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MAXIMAL AND MINIMAL PARTIAL CLONES

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ABSTRACT

The following two problems are addressed in this paper. Let $k \geq 2$, $\mathbf{k}$ be a k-element set and $\mathcal{M}$ be a family of maximal partial clones with trivial intersection over $\mathbf{k}$. What is the smallest possible cardinality of $\mathcal{M}$? Dually, if $\mathcal{F}$ is a family of minimal partial clones whose join is the set of all partial functions on $\mathbf{k}$, then what is the smallest possible cardinality of $\mathcal{F}$? We show that the answer to these two problems is three.

Keywords: Clones, partial clones

1. Introduction

Let $k \geq 2$ and $\mathbf{k} := \{0, \ldots, k-1\}$. For a positive integer n, an n-ary *partial function* on $\mathbf{k}$ is a map $f : \mathrm{dom}(f) \to \mathbf{k}$ where $\mathrm{dom}(f)$ is a subset of $\mathbf{k}^n$, called the *domain* of f. Let $\mathcal{P}_k^{(n)}$ denote the set of all n-ary partial functions on $\mathbf{k}$ and let $\mathcal{P}_k := \bigcup_{n\geq 1} \mathcal{P}_k^{(n)}$. Moreover, set $\mathcal{O}_k^{(n)} := \{f \in \mathcal{P}_k^{(n)} \mid \mathrm{dom}(f) = \mathbf{k}^n\}$ and call $\mathcal{O}_k := \bigcup_{n\geq 1} \mathcal{O}_k^{(n)}$ the set of all *total functions* on the set $\mathbf{k}$.

For $n, m \geq 1$, $f \in \mathcal{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{P}_k^{(m)}$, the *composition* of f and $g_1, \ldots, g_n$, denoted by $f[g_1, \ldots, g_n] \in \mathcal{P}_k^{(m)}$, is the m-ary partial function defined by

$$\mathrm{dom}(f[g_1, \ldots, g_n]) := \{\vec{a} \in \mathbf{k}^m \mid \vec{a} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i),\ (g_1(\vec{a}), \ldots, g_n(\vec{a})) \in \mathrm{dom}(f)\}$$