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INCREASING, NONDECREASING, AND VIRTUALLY CONTINUOUS FUNCTIONS [1]

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ABSTRACT

Increasing and nondecreasing functions, defined on dense subsets of an interval, are examined constructively – that is, using intuitionistic logic. It is shown how the domain of an increasing function can be extended to include all points at which the function is virtually continuous.

Keywords: construction, increasing, nondecreasing, virtually continuous.

Let I be a proper interval in $\mathbb{R}$, and f a real-valued mapping defined on a dense subset D of I. We say that f is

- *nondecreasing* if $\forall x, y \in D\ (x \leq y \Rightarrow f(x) \leqslant f(y))$,
- *increasing* if[2] $\forall x, y \in D\ (f(x) < f(y) \Rightarrow x < y)$, and
- *strictly increasing* if $\forall x, y \in D\ (f(x) < f(y) \Leftrightarrow x < y)$.

In this note we first discuss the relation between *increasing* and *nondecreasing* within Bishop's constructive mathematics (which is, roughly, mathematics with intuitionistic logic).[3] We then introduce the notion of *virtual continuity*, applicable to an increasing function $f : D \to \mathbb{R}$, and show how the domain of f can be extended to include all its points of virtual continuity.

To begin with, let us take the case where $D = I$ and consider the question: *When is a nondecreasing function increasing?*

First, recall *Markov's Principle:*

If (a_n) *is a binary sequence such that* $\neg\forall n\ (a_n = 0)$, *then* $\exists n\ (a_n = 1)$.

[1] Full version of a submission presented at the First Workshop on *Constructivity, Complexity, and Fuzziness* held in Galati, Romania, August 26 – 28, 1999.

[2] Mandelkern [7] calls this notion *antidecreasing*.

[3] Further information about constructive mathematics is found in [1, 2, 3, 5, 9].