Journal of Automata, Languages and Combinatorics 4 (1999) 3, 153–170

DOUBLE SEQUENCES WITH COMPLEXITY $mn + 1$ [1]

JULIEN CASSAIGNE
Institut de Mathématiques de Luminy
Case 907, F-13288 Marseille Cedex 9, France
e-mail: cassaigne@iml.univ-mrs.fr

ABSTRACT

For usual infinite words, a classical result of MORSE and HEDLUND states that if the complexity function satisfies $p(n) \leq n$ for some value of n, then the word is eventually periodic. To generalize this result to two dimensions (in a way which is still to be precised) is an open problem.

We will focus here on the limit case. In one dimension, the non-periodic sequences with lowest possible complexity are Sturmian sequences, for which $p(n) = n + 1$ for all n; their structure has been extensively studied. In two dimensions, it seems that this rôle could be played by double sequences with a rectangle complexity equal to $p(m, n) = mn + 1$. We give an exhaustive description of these sequences, showing that they can be of three different types; in two of these types, one-dimensional Sturmian sequences are used to code the boundary between two parts of the plane.

1. Introduction

The relations between the speed of growth of the complexity function and periodicity, for usual symbolic sequences, are well understood since the work of MORSE and HEDLUND [5]. If a sequence is eventually periodic, then its complexity is bounded, and otherwise its complexity $p(n)$ is at least $n + 1$, which is usually formulated as follows: if there exists an integer n_0 such that $p(n_0) \leq n_0$, then the sequence is eventually periodic. The lower bound $n + 1$ for non-periodic sequences is optimal, since there actually exist non-periodic sequences with a complexity exactly equal to $n + 1$ for all n, namely Sturmian sequences. These sequences, which have been extensively studied (see for instance [2]), have many remarkable properties that make them a fundamental object in combinatorics on words and in symbolic dynamics.

Recently, the question of generalizing these results to higher dimensions has been raised. Many new difficulties appear already in dimension 2, beginning with the choice of the right definitions for periodicity and for the complexity function.

In this article, we will use sequences indexed on $\mathbb{Z}^2$ (in one dimension, sequences considered are usually indexed on $\mathbb{N}$, but we prefer here to use $\mathbb{Z}^2$ instead of $\mathbb{N}^2$ as the choice of an additive group seems to have certain advantages) and with values

[1] Full version of a submission presented at the "Mons' Days of Theoretical Computer Science" at the University of Mons-Hainaut, Belgium, March 2–4, 1998.